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Indecomposability graph and critical vertices of an indecomposable graph

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ABSTRACT

Given a directed graph $G = (V, A)$, the induced subgraph of G by a subset X of V is denoted by $G[X]$. A subset X of V is an interval of G provided that for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$, and similarly for (x, a) and (x, b). For instance, $\emptyset$, V and $\{x\}$, $x \in V$, are intervals of G, called trivial intervals. A directed graph is indecomposable if all its intervals are trivial, otherwise it is decomposable. Given an indecomposable directed graph $G = (V, A)$, a vertex x of G is critical if $G[V \setminus \{x\}]$ is decomposable. An indecomposable directed graph is critical when all its vertices are critical. With each indecomposable directed graph $G = (V, A)$ is associated its indecomposability directed graph $\mathrm{Ind}(G)$ defined on V by: given $x \neq y \in V$, (x, y) is an arc of $\mathrm{Ind}(G)$ if $G[V \setminus \{x, y\}]$ is indecomposable. All the results follow from the study of the connected components of the indecomposability directed graph. First, we prove: if G is an indecomposable directed graph, which admits at least two non critical vertices, then there is $x \in V$ such that $G[V \setminus \{x\}]$ is indecomposable and non critical. Second, we characterize the indecomposable directed graphs G which have a unique non critical vertex x and such that $G[V \setminus \{x\}]$ is critical. Third, we propose a new approach to characterize the critical directed graphs.

1. Introduction

A *(directed) graph* G consists of a finite and nonempty *vertex* set V and an *arc* set A, where an arc is an ordered pair of distinct vertices. Such a graph is denoted by (V, A). Given a graph $G = (V, A)$, with each nonempty subset X of V associate the *subgraph* $G[X] = (X, A \cap (X \times X))$ of G induced by X. Given a proper subset X of V, $G[V \setminus X]$ is also denoted by $G - X$, and by $G - x$ whenever $X = \{x\}$. Given a graph $G = (V, A)$, a nonempty subset C of V is a *connected component* of G provided that for $x \in C$ and $y \in V \setminus C$, $(x, y), (y, x) \notin A$, and for $x \neq y \in C$, there is a sequence $x_0 = x, \dots, x_n = y$ of elements of C satisfying $\{(x_i, x_{i+1}), (x_{i+1}, x_i)\} \cap A \neq \emptyset$ for $0 \leq i \leq n - 1$.

Some notations are needed. Let $G = (V, A)$ be a graph. For $x \neq y \in V$, $x \longrightarrow y$ means $(x, y) \in A$ and $(y, x) \notin A$, $x \longleftrightarrow y$ means $(x, y), (y, x) \in A$ and $x \cdots y$ means $(x, y), (y, x) \notin A$. For $x \in V$ and $Y \subseteq V$, $x \longrightarrow Y$ signifies that for every $y \in Y$, $x \longrightarrow y$. For $X, Y \subseteq V$, $X \longrightarrow Y$ signifies that for every $x \in X$, $x \longrightarrow Y$. For $x \in V$ and for $X, Y \subseteq V$, $x \longleftrightarrow Y$, $x \cdots Y$, $X \longleftrightarrow Y$ and $X \cdots Y$ are defined in the same way. We also introduce an equivalence relation, denoted by $\equiv$, between the ordered pairs of distinct vertices of G. For $x \neq y \in V$ and for $u \neq v \in V$, $(x, y) \equiv (u, v)$ if the function $\{x, y\} \longrightarrow \{u, v\}$, defined by $x \mapsto u$ and $y \mapsto v$, is an isomorphism from $G[\{x, y\}]$ onto $G[\{u, v\}]$. For $Y \subseteq V$ and for $x \in V \setminus Y$, $x \sim Y$ signifies that for any $y, y' \in Y$, $(x, y) \equiv (x, y')$. For $X, Y \subseteq V$, with $X \cap Y = \emptyset$, $X \sim Y$ signifies that for any $x, x' \in X$ and $y, y' \in Y$, $(x, y) \equiv (x', y')$.

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With each graph $G = (V, A)$ associate its *dual* $G^\star = (V, A^\star)$ and its *complement* $\overline{G} = (V, \overline{A})$ defined by: for any $x \neq y \in V$, $(x, y) \in A^\star$ if $(y, x) \in A$, and $(x, y) \in \overline{A}$ if $(x, y) \notin A$. A graph $G = (V, A)$ is *symmetric* if $A = A^\star$. For instance, the path and the cycle are symmetric. Given $n \geq 2$, the *path* P_n is defined on $\{0, \dots, n-1\}$ by: for $i, j \in \{0, \dots, n-1\}$, (i, j) is an arc of P_n if $|i - j| = 1$. The *length* of P_n is $n - 1$. Given $n \geq 3$, the *cycle* C_n is defined on $\{0, \dots, n-1\}$ also. It is obtained from P_n by adding the arcs $(0, n-1)$ and $(n-1, 0)$. The *length* of C_n is n. Let $G = (V, A)$ be a symmetric graph. Given a vertex x of G, a *neighbour* of x is a vertex y of G such that $(x, y), (y, x) \in A$. The family of the neighbours of x is called the *neighbourhood* of x and denoted by $N_G(x)$.

A graph $G = (V, A)$ is a *poset* (or also is *transitive*) provided that for any $x, y, z \in V$, we have: if $(x, y) \in A$ and $(y, z) \in A$, then $(x, z) \in A$. Given a poset $P = (V, A)$, the *comparability graph* of P is the symmetric graph $\mathrm{Comp}(P) = (V, A \cup A^\star)$. Lastly, a graph (V, A) is a *tournament* if for any $x \neq y \in V$, either $x \longrightarrow y$ or $y \longrightarrow x$. A *total order* is both a poset and a tournament. Given a total order $T = (V, A)$, $x < y$ means $x \longrightarrow y$ for $x, y \in V$.

Now, we present the main notion. Given a graph $G = (V, A)$, a subset X of V is an *interval* [3,5] (or a *clan* [4] or a *homogeneous* set [2] or a *module* [7]) of G if for every $x \in V \setminus X$, $x \sim X$. For a tournament $T = (V, A)$, we obtain that a subset X of V is an interval of G if and only if for any $a, b \in X$ and $x \in V$, we have: if $(a, x), (x, b) \in A$, then $x \in X$. This generalizes the classic notion of the interval of a total order. Given a graph $G = (V, A)$, $\emptyset$, V and $\{x\}$, where $x \in V$, are intervals of G, called *trivial* intervals. A graph is *indecomposable* [3,5,6] (or *prime* [2] or *primitive* [4]) if all its intervals are trivial, otherwise it is *decomposable*. Let $G = (V, A)$ be an indecomposable graph. A vertex x of G is *critical* if $G - x$ is decomposable. The family of the critical vertices of G is denoted by $\mathcal{C}(G)$. The graph G is then said to be *critical* if $V = \mathcal{C}(G)$. Similarly, an unordered pair $\{x, y\}$ of distinct vertices of G is critical if $G - \{x, y\}$ is decomposable. Until now, the next three theorems are the most important results concerning the non critical vertices and pairs.

Theorem 1 (*Ehrenfeucht and Rozenberg [4]*). *An indecomposable graph $G = (V, A)$, with $|V| \geq 5$, admits a non critical vertex or a non critical pair.*

Theorem 2 (*Schmerl and Trotter [6]*). *An indecomposable graph $G = (V, A)$, with $|V| \geq 7$, admits a non critical pair.*

The next theorem specifies where non critical pairs occur.

Theorem 3 (*Ille [5]*). *Given an indecomposable graph $G = (V, A)$, if X is a subset of V such that $|X| \geq 3$ and $|V \setminus X| \geq 6$, then $V \setminus X$ contains a non critical pair.*

By examining the connected components of the indecomposability graph associated with any indecomposable graph (see Section 3), we succeed in specifying Theorem 2 in another way.

Theorem 4. *Let $G = (V, A)$ be an indecomposable graph with $|V| \geq 7$. If $|V \setminus \mathcal{C}(G)| \geq 2$, then G admits a non critical pair which intersects $V \setminus \mathcal{C}(G)$.*

The following equivalent statement indicates how the property that G is indecomposable without being critical is hereditary. *Let $G = (V, A)$ be a non critical and indecomposable graph with $|V| \geq 7$. If $|V \setminus \mathcal{C}(G)| \geq 2$, then G admits a vertex x such that $G - x$ is indecomposable and non critical as well.* Then, we characterize the indecomposable graphs G which possess a single non critical vertex x and such that $G - x$ is critical.

Finally, we provide a new presentation of the critical graphs which follows from the examination of the connected components of the indecomposability graph.

2. The indecomposable graphs

We review the indecomposable subgraphs of an indecomposable graph. We begin with the indecomposable subgraphs of small cardinality.

Lemma 5 (*Sumner [8]*). *Given an indecomposable graph $G = (V, A)$, with $|V| \geq 3$, there is a subset X of V such that $|X| = 3$ or 4 and $G[X]$ is indecomposable.*

Proposition 6 (*Cournier and Ille [3]*). *Let $G = (V, A)$ be an indecomposable graph, with $|V| \geq 3$. For each $x \in V$, there exists a subset X of V such that $x \in X$, $3 \leq |X| \leq 5$ and $G[X]$ is indecomposable.*

To construct indecomposable subgraphs of a larger size, we use the partition below. Let $G = (V, A)$ be a graph. Given a proper subset X of V such that $|X| \geq 3$ and $G[X]$ is indecomposable, consider the following subsets of $V \setminus X$.

- $\mathrm{Ext}(X)$ is the set of $x \in V \setminus X$ such that $G[X \cup \{x\}]$ is indecomposable;
- $\langle X \rangle$ is the set of $x \in V \setminus X$ such that X is an interval of $G[X \cup \{x\}]$;
- for each $u \in X$, $X(u)$ is the set of $x \in V \setminus X$ such that $\{u, x\}$ is an interval of $G[X \cup \{x\}]$.

The family constituted by $\mathrm{Ext}(X)$, $\langle X \rangle$ and $X(u)$, where $u \in X$, is denoted by p_X. It realizes a partition of $V \setminus X$.

Lemma 7 (*Ehrenfeucht and Rozenberg [4]*). *Given an indecomposable graph $G = (V, A)$, if X is a subset of V such that $|X| \geq 3$, $|V \setminus X| \geq 2$ and $G[X]$ is indecomposable, then there are $x \neq y \in V \setminus X$ such that $G[X \cup \{x, y\}]$ is indecomposable.*

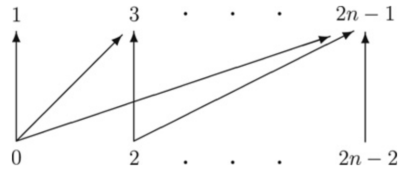

Fig. 1. Q_{2n}.

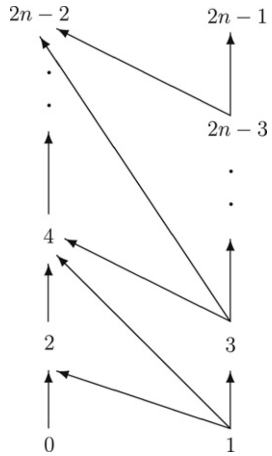

Fig. 2. R_{2n}.

Theorem 1 is a consequence of Lemma 5 and 7. Theorem 2 is established from the characterization of the critical graphs due to Schmerl and Trotter [6]. In the particular case of the posets, they obtained the following two posets Q_{2n} and R_{2n} defined on $\{0, \dots, 2n-1\}$, where $n \geq 2$ (see Figs. 1 and 2).

- For any $x \neq y \in \{0, \dots, 2n-1\}$, (x, y) is an arc of Q_{2n} if there are $0 \leq i \leq j \leq n-1$ such that $(x, y) = (2i, 2j+1)$.
- For any $x \neq y \in \{0, \dots, 2n-1\}$, (x, y) is an arc of R_{2n} if $x < y$ and if x is odd or y is even.

Remark 8. Consider an integer $n \geq 3$. For every $i \in \{1, \dots, 2n-2\}$, we obtain that $\{i-1, i+1\}$ is an interval of $Q_{2n} - i$. Furthermore, $\{2, \dots, 2n-1\}$ is an interval of $Q_{2n} - 0$ and $\{0, \dots, 2n-3\}$ is an interval of $Q_{2n} - (2n-1)$. It follows that for any $i \neq j \in \{0, \dots, 2n-1\}$, the three assertions below are equivalent:

- $Q_{2n} - \{i, j\}$ is isomorphic to Q_{2n-2};
- $Q_{2n} - \{i, j\}$ is indecomposable;
- $|i - j| = 1$.

Lastly, notice that Q_{2n} and $(Q_{2n})^\star$ are isomorphic by considering the permutation of $\{0, \dots, 2n-1\}$ defined by $i \mapsto (2n-1) - i$. The same observations hold for R_{2n}.

Theorem 9 (*Schmerl and Trotter [6]*). *Given an indecomposable poset $P = (V, A)$, with $|V| \geq 4$, P is critical if and only if $|V|$ is even and P is isomorphic to $Q_{|V|}$ or $R_{|V|}$.*

3. The indecomposability graph

With each indecomposable graph $G = (V, A)$ associate its *indecomposability graph* $\mathrm{Ind}(G)$ defined on V by: given $x \neq y \in V$, (x, y) is an arc of $\mathrm{Ind}(G)$ if $\{x, y\}$ is a non critical pair of G. For instance, for $n \geq 3$, we have $\mathrm{Ind}(Q_{2n}) = \mathrm{Ind}(R_{2n}) = P_{2n}$ by Remark 8. To begin, we examine the neighbourhood $N_{\mathrm{Ind}(G)}(x)$ of a critical vertex x of an indecomposable graph G.

Lemma 10. *Let $G = (V, A)$ be an indecomposable graph with $|V| \geq 5$. For every $x \in \mathcal{C}(G)$, $|N_{\mathrm{Ind}(G)}(x)| \leq 2$. Moreover, we have:*

(1) *if $|N_{\mathrm{Ind}(G)}(x)| = 1$, then $V \setminus (N_{\mathrm{Ind}(G)}(x) \cup \{x\})$ is an interval of $G - x$;*
(2) *if $|N_{\mathrm{Ind}(G)}(x)| = 2$, then $N_{\mathrm{Ind}(G)}(x)$ is an interval of $G - x$.*

Proof. To begin, we prove that $|N_{\mathrm{Ind}(G)}(x)| \leq 2$ for each $x \in \mathcal{C}(G)$. Consider $x \in \mathcal{C}(G)$ such that $N_{\mathrm{Ind}(G)}(x) \neq \emptyset$. Given $y \in N_{\mathrm{Ind}(G)}(x)$, denote $V \setminus \{x, y\}$ by X. Since $G - x$ is decomposable, $y \notin \mathrm{Ext}(X)$. We distinguish the following two cases.

(a) Assume that $y \in \langle X \rangle$. For every $z \in X$, $X \setminus \{z\}$ is a non trivial interval of $G - \{x, z\}$. Therefore, $z \notin N_{\mathrm{Ind}(G)}(x)$ and hence $N_{\mathrm{Ind}(G)}(x) = \{y\}$.
(b) Assume that there is $u \in X$ such that $y \in X(u)$. For every $z \in X \setminus \{u\}$, $\{u, y\}$ is a non trivial interval of $G - \{x, z\}$. Consequently, $z \notin N_{\mathrm{Ind}(G)}(x)$ and thus $N_{\mathrm{Ind}(G)}(x) \subseteq \{y, u\}$. As $\{u, y\}$ is an interval of $G[X \cup \{y\}]$, the function $X \longrightarrow (X \setminus \{u\}) \cup \{y\}$, defined by $u \mapsto y$ and $v \mapsto v$ for $v \in X \setminus \{u\}$, is an isomorphism from $G - \{x, y\}$ onto $G - \{x, u\}$. It follows that $u \in N_{\mathrm{Ind}(G)}(x)$ so that $N_{\mathrm{Ind}(G)}(x) = \{y, u\}$.

It results from the two cases that $|N_{\mathrm{Ind}(G)}(x)| \leq 2$. Now, assume that $|N_{\mathrm{Ind}(G)}(x)| = 1$ and denote by y the unique element of $N_{\mathrm{Ind}(G)}(x)$. It follows from (b) that $y \notin X(u)$ for every $u \in X$. Consequently, $y \in \langle X \rangle$ or, equivalently, $V \setminus \{x, y\}$ is an interval of $G - x$. Lastly, assume that $|N_{\mathrm{Ind}(G)}(x)| = 2$ and denote the elements of $N_{\mathrm{Ind}(G)}(x)$ by y and z. It follows from (a) that $y \notin \langle X \rangle$ and thus there is $u \in X$ such that $y \in X(u)$. By (b), $u \in N_{\mathrm{Ind}(G)}(x)$ and, necessarily, $u = z$. So, $\{y, z\}$ is an interval of $G - x$. $\square$

Given an indecomposable graph G, consider a connected component C of $\mathrm{Ind}(G)$ such that $|C| \geq 2$ and $C \subseteq \mathcal{C}(G)$. It follows from the preceding lemma that $\mathrm{Ind}(G)[C]$ is a cycle or a path.

Proposition 11. *Let $G = (V, A)$ be an indecomposable graph with $|V| \geq 5$. For every connected component C of $\mathrm{Ind}(G)$ such that $|C| \geq 2$ and $C \subseteq \mathcal{C}(G)$, we have:*

(1) *If $\mathrm{Ind}(G)[C]$ is a cycle, then its length is odd and $C = V$;*
(2) *If $\mathrm{Ind}(G)[C]$ is a path of odd length, then $|V \setminus C| \leq 1$;*
(3) *If $\mathrm{Ind}(G)[C]$ is a path of even length, then $C = V$.*

Proof. We denote the elements of C by $0, \ldots, l - 1$ in such a way that $\mathrm{Ind}(G)[C] = C_l$ or P_l.

Firstly, suppose that $l \geq 3$ and $\mathrm{Ind}(G)[C] = C_l$. We prove that if there is $n \geq 2$ such that $l = 2n$, then $\{1, 2n - 1\}$ would be an interval of G. As $N_{\mathrm{Ind}(G)}(0) = \{1, 2n - 1\}$, $\{1, 2n - 1\}$ is an interval of $G - 0$ by Lemma 10. It suffices to verify that $(0, 1) \equiv (0, 2n - 1)$. For $i \in \{1, \ldots, n - 1\}$, we have $N_{\mathrm{Ind}(G)}(2i) = \{2i - 1, 2i + 1\}$. By Lemma 10, $\{2i - 1, 2i + 1\}$ is an interval of $G - (2i)$ and, in particular, $(0, 2i - 1) \equiv (0, 2i + 1)$. Therefore, $(0, 1) \equiv (0, 3) \equiv \cdots \equiv (0, 2n - 1)$. It follows that there is $n \geq 1$ such that $l = 2n + 1$. For a contradiction, suppose that $C \neq V$ and consider an element x of $V \setminus C$. For $i \in \{0, \ldots, n-1\}$, we have $N_{\mathrm{Ind}(G)}(2i+1) = \{2i, 2i+2\}$. By Lemma 10, $\{2i, 2i+2\}$ is an interval of $G-(2i+1)$. In particular, we obtain that $(x, 0) \equiv (x, 2) \equiv \cdots \equiv (x, 2n)$. Similarly, since for $i \in \{1, \ldots, n - 1\}$, $N_{\mathrm{Ind}(G)}(2i) = \{2i - 1, 2i + 1\}$, we have $x \sim \{2j + 1; 0 \leq j \leq n - 1\}$. As $N_{\mathrm{Ind}(G)}(0) = \{1, 2n\}$, $(x, 2n) \equiv (x, 1)$ and hence $x \sim C$. Consequently, C would be an interval of G. It follows that $C = V$.

Secondly, assume that $\mathrm{Ind}(G)[C] = P_l$ and $l = 2n \geq 2$ is even. If $n = 1$, then $N_{\mathrm{Ind}(G)}(0) = \{1\}$ and $N_{\mathrm{Ind}(G)}(1) = \{0\}$. By Lemma 10, $V \setminus \{0, 1\}$ is an interval of $G - 0$ and of $G - 1$. Thus, $V \setminus \{0, 1\}$ would be a non trivial interval of G. Therefore, $n > 1$. To conclude, it is sufficient to show that $V \setminus C$ is an interval of G. Consider an element x of $V \setminus X$. Since for $i \in \{0, \ldots, n-2\}$, $N_{\mathrm{Ind}(G)}(2i+1) = \{2i, 2i+2\}$, we have $x \sim \{2j; 0 \leq j \leq n-1\}$. Moreover, as $N_{\mathrm{Ind}(G)}(2n-1) = \{2n-2\}$, $(x, 2n-2) \equiv (1, 2n-2)$. It follows that for $y, z \in V \setminus C$ and $j \in \{0, \ldots, n - 1\}$, $(y, 2j) \equiv (z, 2j)$. Similarly, $x \sim \{2j + 1; 0 \leq j \leq n - 1\}$ because $N_{\mathrm{Ind}(G)}(2i) = \{2i - 1, 2i + 1\}$ for $i \in \{1, \ldots, n - 1\}$. But, since $N_{\mathrm{Ind}(G)}(0) = \{1\}$, $(x, 1) \equiv (2n - 2, 1)$. Consequently, for $y, z \in V \setminus C$ and $j \in \{0, \ldots, n - 1\}$, $(y, 2j + 1) \equiv (z, 2j + 1)$.

Finally, assume that $\mathrm{Ind}(G)[C] = P_l$ and $l = 2n + 1 \geq 3$ is odd. We establish that if C is a proper subset of V, then $V \setminus \{1\}$ would be an interval of G. As $N_{\mathrm{Ind}(G)}(0) = \{1\}$, $V \setminus \{0, 1\}$ is an interval of $G - 0$ by Lemma 10. Consequently, we have to verify that $(1, x) \equiv (1, 0)$, where $x \in V \setminus C$. If $n = 1$, then $N_{\mathrm{Ind}(G)}(2) = \{1\}$. By Lemma 10, $V \setminus \{1, 2\}$ is an interval of $G - 2$ and, in particular, $(1, x) \equiv (1, 0)$. Assume that $n > 1$. Since for $i \in \{1, \ldots, n - 1\}$, $N_{\mathrm{Ind}(G)}(2i) = \{2i - 1, 2i + 1\}$, we have $x \sim \{2j + 1; 0 \leq j \leq n - 1\}$ and $0 \sim \{2j + 1; 0 \leq j \leq n - 1\}$. Therefore, $(1, x) \equiv (2n - 1, x)$ and $(1, 0) \equiv (2n - 1, 0)$. As $N_{\mathrm{Ind}(G)}(2n) = \{2n - 1\}$, $(2n - 1, x) \equiv (2n - 1, 0)$. $\square$

We immediately obtain:

Corollary 12. *Let $G = (V, A)$ be an indecomposable graph with $|V| \geq 5$ and $|V \setminus \mathcal{C}(G)| \geq 2$. A connected component C of $\mathrm{Ind}(G)$ such that $|C| \geq 2$ is not contained in $\mathcal{C}(G)$.*

A simple proof of Theorem 4 follows.

Proof of Theorem 4. By Theorem 2, there exist distinct vertices x and y such that $G - \{x, y\}$ is indecomposable. Denote by C the connected component of $\mathrm{Ind}(G)$ which contains x and y. By Corollary 12, C intersects $V \setminus \mathcal{C}(G)$. Consider $\alpha \in C \cap (V \setminus \mathcal{C}(G))$. As $\mathrm{Ind}(G)[C]$ is connected there is $\beta \in C$ such that (α, β) is an arc of $\mathrm{Ind}(G)$. Thus, $\{\alpha, \beta\}$ is a non critical pair of G which intersects $V \setminus \mathcal{C}(G)$. $\square$

To envisage the case where only one non critical vertex exists, we use the extension R_{2n+1} of R_{2n} to $\{0, \ldots, 2n\}$, where $n \geq 2$, defined by $R_{2n+1}[\{0, \ldots, 2n - 1\}] = R_{2n}$ and $\{1, 3, \ldots, 2n - 1\} \longrightarrow 2n \longrightarrow \{0, 2, \ldots, 2n - 2\}$.

Remark 13. Given $n \geq 2$, we verify that R_{2n+1} is indecomposable by using the partition p_X for $X = \{0, \ldots, 2n - 1\}$, since $R_{2n+1} - (2n) = R_{2n}$ is indecomposable. It follows from Remark 8 that $\mathcal{C}(R_{2n+1}) = \{0, \ldots, 2n - 1\}$, $\mathrm{Ind}(R_{2n+1}) - (2n) = P_{2n}$ and $N_{\mathrm{Ind}(R_{2n+1})}(2n) = \emptyset$. We also have that R_{2n+1} and $(R_{2n+1})^\star$ are isomorphic by considering the permutation of $\{0, \ldots, 2n\}$ defined by $2n \mapsto 2n$ and $i \mapsto (2n - 1) - i$ for $i \in \{0, \ldots, 2n - 1\}$.

Theorem 14. *Let $G = (V, A)$ be an indecomposable graph with $|V| \geq 7$ and $|V \setminus \mathcal{C}(G)| = 1$. The indecomposability graph of G admits a unique connected component C such that $|C| \geq 2$. Moreover, if $C \subseteq \mathcal{C}(G)$, then G is isomorphic to R_{2n+1} or to $\overline{R_{2n+1}}$.*

Proof. By Proposition 6, there exists a subset X of V such that $3 \leq |X| \leq 5$, $G[X]$ is indecomposable and X contains the unique non critical vertex of G. By Lemma 7 applied several times from $G[X]$, we obtain elements x and y of $V \setminus X$ such that $G - \{x, y\}$ is indecomposable. As $V \setminus X \subseteq \mathcal{C}(G)$, we have $x \neq y$. Denote by C the connected component of $\mathrm{Ind}(G)$ which contains x and y. If $\mathrm{Ind}(G)$ admits at least two connected components which are not reduced to a singleton, then one of them should be contained in $\mathcal{C}(G)$, which contradicts Proposition 11. Consequently, C is the unique connected component of $\mathrm{Ind}(G)$ such that $|C| \geq 2$. At present, assume that $C \subseteq \mathcal{C}(G)$. By Proposition 11, C is a path of odd length and $|V \setminus C| = 1$. Let us denote the vertices of G by $0, \ldots, 2n$ in such a way that $C = \{0, \ldots, 2n-1\}$, $\mathrm{Ind}(G)[C] = P_{2n}$, $V \setminus \mathcal{C}(G) = \{2n\}$ and $N_{\mathrm{Ind}(G)}(2n) = \emptyset$. We verify that G is entirely determined by $G[\{0, 2n-1\}]$ and $G[\{0, 2n\}]$ by using Lemma 10. Given $i \in \{0, \ldots, n-2\}$: as $N_{\mathrm{Ind}(G)}(2i+1) = \{2i, 2i+2\}$, $\{2i, 2i+2\}$ is an interval of $G - (2i+1)$ and, in particular, $(2n, 2i) \equiv (2n, 2i+2)$. Therefore:

$$2n \sim \{2i; 0 \leq i \leq n-1\}. \tag{1}$$

As $N_{\mathrm{Ind}(G)}(0) = \{1\}$, $1 \sim \{2, \ldots, 2n\}$ and as $N_{\mathrm{Ind}(G)}(2n-1) = \{2n-2\}$, $2n-2 \sim \{0, \ldots, 2n-3\} \cup \{2n\}$. By using (1), for $x \in \{2, \ldots, 2n\}$ and $y \in \{0, \ldots, 2n-3\}$, we have:

$$(1, x) \equiv (1, 2n-2) \equiv (y, 2n-2) \equiv (2n, 2n-2) \equiv (2n, 0). \tag{2}$$

Given $1 \leq i \leq n-1$: as $N_{\mathrm{Ind}(G)}(2i) = \{2i-1, 2i+1\}$, $\{2i-1, 2i+1\}$ is an interval of $G - (2i)$ and, in particular, $(2n, 2i-1) \equiv (2n, 2i+1)$. By using (2), we have:

$$(2n, 2n-1) \equiv (2n, 2n-3) \equiv \cdots \equiv (2n, 1) \equiv (0, 2n). \tag{3}$$

Now, consider $x < y \in \{0, \ldots, 2n-1\}$. Assume that $x \geq 2$. Since $N_{\mathrm{Ind}(G)}(x-1) = \{x-2, x\}$, $\{x-2, x\}$ is an interval of $G - (x-1)$. In particular, we have $(x-2, y) \equiv (x, y)$. Similarly, if $y \leq 2n-3$, then $(x, y) \equiv (x, y+2)$ because $N_{\mathrm{Ind}(G)}(y+1) = \{y, y+2\}$. Therefore, we obtain for $0 \leq i < j \leq n-1$:

$$\left.\begin{array}{r}(2i, 2j) \equiv (0, 2j) \equiv (0, 2n-2) \equiv (2n, 0)\\ (2i, 2j-1) \equiv (0, 2j-1) \equiv (0, 2n-1)\\ (2i+1, 2j+1) \equiv (1, 2j+1) \equiv (1, 2n-1) \equiv (2n, 0)\\ (2i+1, 2j) \equiv (1, 2j) \equiv (1, 2n-2) \equiv (2n, 0)\end{array}\right\}. \tag{4}$$

It follows from (1) and (3) that $(0, 2n) \not\equiv (2n, 0)$. By interchanging G and its complement $\overline{G}$, assume that $2n \longrightarrow 0$. By (4), $G[\{0, 2, \ldots, 2n-2\}]$ is the usual total order on $\{0, 2, \ldots, 2n-2\}$, $G[\{1, 3, \ldots, 2n-1\}]$ is the usual total order on $\{1, 3, \ldots, 2n-1\}$ and for $0 \leq i < j \leq n-1$, $2i+1 \longrightarrow 2j$. Therefore, if $0 \longrightarrow 2n-1$, then $G - (2n)$ is the usual total order on $\{0, \ldots, 2n-1\}$. Similarly, if $2n-1 \longrightarrow 0$, then $G - (2n)$ is the total order: $1 < 3 < \cdots < 2n-1 < 0 < 2 < \cdots < 2n-2$. Consequently, if $(0, 2n-1) \not\equiv (2n-1, 0)$, then $G - (2n)$ is a total order and hence $G - (2n)$ is decomposable. Since $2n$ is a non critical vertex of G, we obtain that $(0, 2n-1) \equiv (2n-1, 0)$. Thus, either $0 \cdots 2n-1$ and $G = R_{2n+1}$ or $0 \longleftrightarrow 2n-1$ and $\overline{G} = (R_{2n+1})^{\star}$. In the last instance, G is isomorphic to $\overline{R_{2n+1}}$ by Remark 13. □

Unlike for Theorem 4, we do not use Theorem 2 in the preceding proof. Theorem 14 leads to the following problem.

Problem 15. *Characterize the indecomposable graphs which admit a unique non critical vertex.*

The following consequence of Corollary 12 and Theorem 14 is satisfied by the posets, the symmetric graphs, the tournaments…

Corollary 16. *Let $G = (V, A)$ be a non critical and indecomposable graph, with $|V| \geq 7$, which is isomorphic neither to R_{2n+1} nor to $\overline{R_{2n+1}}$. Then, a connected component C of $\mathrm{Ind}(G)$ exists such that $|C| \geq 2$ and is not contained in $\mathcal{C}(G)$. In particular, G possesses a vertex x such that $G - x$ is indecomposable and non critical as well.*

Consider an indecomposable graph $G = (V, A)$, such that $|V| \geq 7$, which possesses a unique non critical vertex x and which is isomorphic neither to R_{2n+1} nor to $\overline{R_{2n+1}}$. By the preceding corollary, $G - x$ admits at least one non critical vertex or equivalently $N_{\mathrm{Ind}(G)}(x) \neq \emptyset$. For each $n \geq 3$, Belkhechine, Boudabbous and Elayech [1] defined the symmetric graph Γ_{2n+1} on $\{0, \ldots, 2n\}$ as follows. For any $i \neq j \in \{0, \ldots, 2n\}$, (i, j) is an arc of Γ_{2n+1} if either i and j are even or there is $k \in \{0, \ldots, n-1\}$ such that $\{i, j\} = \{2k+1, 2k+2\}$. They showed that Γ_{2n+1} is indecomposable, $\mathcal{C}(\Gamma_{2n+1}) = \{1, \ldots, 2n\}$ and $N_{\mathrm{Ind}(\Gamma_{2n+1})}(0) = \{2k+1; 0 \leq k \leq n-1\}$.

4. A new presentation of the critical graphs

We commence with a direct consequence of Proposition 11.

Corollary 17. *If $G = (V, A)$ is a critical graph, with $|V| \geq 5$, then $\mathrm{Ind}(G)$ satisfies one of the following:*

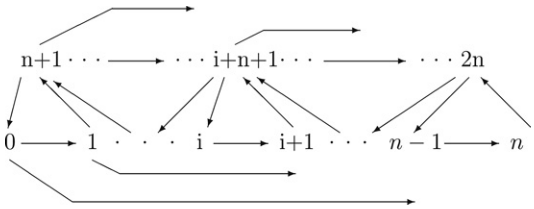

Fig. 3. T_{2n+1}.

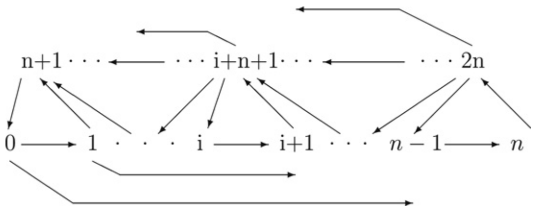

Fig. 4. U_{2n+1}.

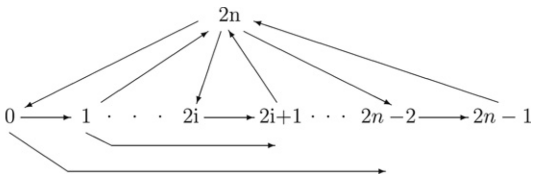

Fig. 5. W_{2n+1}.

(1) $\mathrm{Ind}(G)$ *is a cycle of odd length;*
(2) $\mathrm{Ind}(G)$ *is a path;*
(3) $|V|$ *is odd and there is* $x \in V$ *such that* $\mathrm{Ind}(G) - x$ *is a path and* $N_{\mathrm{Ind}(G)}(x) = \emptyset$.

Proof. By Lemma 5, there exists a subset X of V such that $|X| = 3$ or 4 and $G[X]$ is indecomposable. By Lemma 7 applied several times from $G[X]$, we obtain elements x and y of $V \setminus X$ such that $G - \{x, y\}$ is indecomposable. Since G is critical, we have $x \neq y$. It is then sufficient to apply Proposition 11 to the connected component of $\mathrm{Ind}(G)$ which contains x and y. $\square$

To review all the critical graphs, we need the tournaments T_{2n+1}, U_{2n+1} and W_{2n+1} defined on $\{0, \ldots, 2n\}$, where $n \geq 1$, as follows (see Figs. 3–5):

- $T_{2n+1}[\{0, \ldots, n\}]$ is the usual total order on $\{0, \ldots, n\}$, $T_{2n+1}[\{n+1, \ldots, 2n\}]$ is the usual order on $\{n+1, \ldots, 2n\}$ and for every $i \in \{0, \ldots, n-1\}$, $\{i+1, \ldots, n\} \longrightarrow i+n+1 \longrightarrow \{0, \ldots, i\}$;
- U_{2n+1} is obtained from T_{2n+1} by reversing all the arcs contained in $\{n+1, \ldots, 2n\}$;
- $W_{2n+1}[\{0, \ldots, 2n-1\}]$ is the usual total order on $\{0, \ldots, 2n-1\}$ and $\{1, 3, \ldots, 2n-1\} \longrightarrow 2n \longrightarrow \{0, 2, \ldots, 2n-2\}$.

We also consider the graph H_{2n+1} defined on $\{0, \ldots, 2n\}$, where $n \geq 1$, by: given $i, j \in \{0, \ldots, 2n\}$, (i, j) is an arc of H_{2n+1} if $i < j$ and if i and j are not both even. In the four propositions below, we examine the critical graphs according to their indecomposability graph characterized in Corollary 17. In the first two, f_{2n+1} denotes the permutation of $\mathbb{Z}_{2n+1}$ defined by $f_{2n+1}(x) = (n+1) \times x \pmod{2n+1}$.

Proposition 18. *Let* $G = (V, A)$ *be a critical graph such that* $|V| \geq 5$. *Given* $n \geq 2$, *if* $\mathrm{Ind}(G) = f_{2n+1}(C_{2n+1})$, *then* $G = T_{2n+1}$ *or* $(T_{2n+1})^\star$.

Proof. To begin, we prove that the permutation θ of $\mathbb{Z}_{2n+1}$, defined by $\theta(x) = x + 1 \pmod{2n+1}$, is an automorphism of G. It suffices to verify that for any $x < y \in \{0, \ldots, 2n\}$, $(x, y) \equiv (x+1, y+1)$. We distinguish the three cases below.

- Assume that $y \notin \{x+1, x+n+1\}$. As $N_{\mathrm{Ind}(G)}(x+n+1) = \{x, x+1\}$, $\{x, x+1\}$ is an interval of $G-(x+n+1)$ by Lemma 10. In particular, we have $(x, y) \equiv (x+1, y)$. Similarly, since $x+1 \notin \{y, y+1, y+n+1\}$ and since $N_{\mathrm{Ind}(G)}(y+n+1) = \{y, y+1\}$, $(x+1, y) \equiv (x+1, y+1)$.
- Assume that $y = x+1$. As $N_{\mathrm{Ind}(G)}(x+n+2) = \{x+1, x+2\}$, we have $(x, x+1) \equiv (x, x+2)$. But, $(x, x+2) \equiv (x+1, x+2)$ because $N_{\mathrm{Ind}(G)}(x+n+1) = \{x, x+1\}$.
- Assume that $y = x+n+1$. Since $N_{\mathrm{Ind}(G)}(x+1) = \{x+n+1, x+n+2\}$, we have $(x, x+n+1) \equiv (x, x+n+2)$. But, $(x, x+n+2) \equiv (x+1, x+n+2)$ because $N_{\mathrm{Ind}(G)}(x+n+1) = \{x, x+1\}$.

It follows that for any $x < y \in \{0, \ldots, 2n\}$, $(x, y) \equiv (0, y - x)$. For $1 \leq i \leq n - 1$, $N_{\mathrm{Ind}(G)}(i + n + 1) = \{i, i + 1\}$. By Lemma 10, $\{i, i + 1\}$ is an interval of $G - (i + n + 1)$ and, in particular, $(0, i) \equiv (0, i + 1)$. Consequently, $0 \sim \{1, 2, \ldots, n\}$. Similarly, $0 \sim \{n + 1, n + 2, \ldots, 2n\}$ because $N_{\mathrm{Ind}(G)}(i + n + 1) = \{i, i + 1\}$ for $n + 1 \leq i \leq 2n - 1$. Furthermore, we obtain by adding n that $(0, n + 1) \equiv (n, 0)$ and we proved that $(n, 0) \equiv (1, 0)$. It follows that for any $x < y \in \{0, \ldots, 2n\}$, we have $(x, y) \equiv (0, 1)$ if $y - x \in \{1, \ldots, n\}$, and $(x, y) \equiv (1, 0)$ if $y - x \in \{n + 1, \ldots, 2n\}$. Since G is indecomposable, $(0, 1) \not\equiv (1, 0)$. To conclude, it suffices to verify that $G = T_{2n+1}$ when $0 \longrightarrow 1$. □

Proposition 19. *Let $G = (V, A)$ be a critical graph such that $|V| \geq 5$. Given $n \geq 2$, if $\mathrm{Ind}(G) = P_{2n+1}$, then G or $G^\star = (f_{2n+1})^{-1}(U_{2n+1})$, H_{2n+1} or $\overline{H_{2n+1}}$.*

Proof. Consider $x < y \in \{0, \ldots, 2n\}$. Assume that $x \geq 2$. As $N_{\mathrm{Ind}(G)}(x - 1) = \{x - 2, x\}$, $\{x - 2, x\}$ is an interval of $G - (x - 1)$ by Lemma 10. In particular, we have $(x, y) \equiv (x - 2, y)$. Similarly, if $y \leq 2n - 2$, then $(x, y) \equiv (x, y + 2)$. Following the parity of x and of y, we obtain four cases.

(1) If x and y are even, then $(x, y) \equiv (0, 2n)$.
(2) If x and y are odd, then $(x, y) \equiv (1, 2n - 1)$.
(3) If x is even and y is odd, then $(x, y) \equiv (0, 2n - 1)$. By Lemma 10, $2n - 1 \sim \{0, \ldots, 2n - 2\}$ since $N_{\mathrm{Ind}(G)}(2n) = \{2n - 1\}$. In particular, we have $(0, 2n - 1) \equiv (1, 2n - 1)$ and thus $(x, y) \equiv (1, 2n - 1)$.
(4) If x is odd and y is even, then $(x, y) \equiv (1, 2n)$. As $N_{\mathrm{Ind}(G)}(0) = \{1\}$, it follows from Lemma 10 that $1 \sim \{2, \ldots, 2n\}$. In particular, we get $(1, 2n) \equiv (1, 2n - 1)$ and hence $(x, y) \equiv (1, 2n - 1)$.

It follows from (3) and (4) that $(0, 1) \equiv (1, 2n - 1)$ and $(2, 1) \equiv (2n - 1, 1)$. By Lemma 10, $\{2, \ldots, 2n\}$ is an interval of $G - 0$ because $N_{\mathrm{Ind}(G)}(0) = \{1\}$. As G is indecomposable, $\{2, \ldots, 2n\}$ is not an interval of G. Therefore, $(0, 1) \not\equiv (2, 1)$ and thus $(1, 2n - 1) \not\equiv (2n - 1, 1)$. By interchanging G and $G^\star$, assume that $1 \longrightarrow 2n - 1$. To conclude, we distinguish two cases.

- Assume that $(0, 2n) \not\equiv (2n, 0)$. Observe that if $0 \longrightarrow 2n$, then G is the usual total order on $\{0, \ldots, 2n\}$. Since G is indecomposable, we have $2n \longrightarrow 0$ and we verify that $f_{2n+1}(G) = (U_{2n+1})^\star$.
- Assume that $(0, 2n) \equiv (2n, 0)$. If $0 \cdots 2n$, then $G = H_{2n+1}$, and if $0 \longleftrightarrow 2n$, then $G = (\overline{H_{2n+1}})^\star$. □

Proposition 20. *Let $G = (V, A)$ be a critical graph such that $|V| \geq 6$. Given $n \geq 3$, if $\mathrm{Ind}(G) = P_{2n}$, then G or $\overline{G} = \mathrm{Comp}(Q_{2n})$, Q_{2n}, $(Q_{2n})^\star$, R_{2n} or $(R_{2n})^\star$.*

Proof. As in the preceding proof, we obtain four cases for $x < y \in \{0, \ldots, 2n - 1\}$.

(1) If x and y are even, then $(x, y) \equiv (0, 2n - 2)$. By Lemma 10, $2n - 2 \sim \{0, \ldots, 2n - 3\}$ since $N_{\mathrm{Ind}(G)}(2n - 1) = \{2n - 2\}$. In particular, we have $(0, 2n - 2) \equiv (1, 2n - 2)$ and thus $(x, y) \equiv (1, 2n - 2)$.
(2) If x and y are odd, then $(x, y) \equiv (1, 2n - 1)$. By Lemma 10, $1 \sim \{2, \ldots, 2n - 1\}$ because $N_{\mathrm{Ind}(G)}(0) = \{1\}$. In particular, we obtain that $(1, 2n - 1) \equiv (1, 2n - 2)$ and hence $(x, y) \equiv (1, 2n - 2)$.
(3) If x is odd and y is even, then $(x, y) \equiv (1, 2n - 2)$.
(4) If x is even and y is odd, then $(x, y) \equiv (0, 2n - 1)$.

It follows from (3) and (4) that $(2, 1) \equiv (2n - 2, 1)$ and $(0, 1) \equiv (0, 2n - 1)$. By Lemma 10, $\{2, \ldots, 2n - 1\}$ is an interval of $G - 0$ because $N_{\mathrm{Ind}(G)}(0) = \{1\}$. As G is indecomposable, $\{2, \ldots, 2n - 1\}$ is not an interval of G. Therefore, $(0, 1) \not\equiv (2, 1)$ and hence $(0, 2n - 1) \not\equiv (2n - 2, 1)$. Finally, we distinguish two cases.

- Assume that $(1, 2n - 2) \not\equiv (2n - 2, 1)$ and, by interchanging G and $G^\star$, that $1 \longrightarrow 2n - 2$. For a contradiction, suppose that $(0, 2n - 1) \not\equiv (2n - 1, 0)$. As $(0, 2n - 1) \not\equiv (2n - 2, 1)$, we get $0 \longrightarrow 2n - 1$. Thus, G would be the usual total order on $\{0, \ldots, 2n - 1\}$, which contradicts the fact that G is indecomposable. Consequently, $(0, 2n - 1) \equiv (2n - 1, 0)$. So, either $0 \cdots 2n - 1$ and $G = R_{2n}$ or $0 \longleftrightarrow 2n - 1$ and $G = (\overline{R_{2n}})^\star$.
- Assume that $(1, 2n - 2) \equiv (2n - 2, 1)$ and, by interchanging G and $\overline{G}$, that $1 \cdots 2n - 2$. If $(0, 2n - 1) \equiv (2n - 1, 0)$, then $0 \longleftrightarrow 2n - 1$ because $(0, 2n - 1) \not\equiv (2n - 2, 1)$. We obtain that $G = \mathrm{Comp}(Q_{2n})$. Lastly, if $(0, 2n - 1) \not\equiv (2n - 1, 0)$, then either $0 \longrightarrow 2n - 1$ and $G = Q_{2n}$ or $2n - 1 \longrightarrow 0$ and $G = (Q_{2n})^\star$. □

Proposition 21. *Let $G = (V, A)$ be a critical graph such that $V = \{0, \ldots, 2n\}$, where $n \geq 2$. If $\mathrm{Ind}(G) - (2n) = P_{2n}$ and $N_{\mathrm{Ind}(G)}(2n) = \emptyset$, then $G = W_{2n+1}$ or $(W_{2n+1})^\star$.*

Proof. As in the proof of Proposition 20, we obtain for $x < y \in \{0, \ldots, 2n - 1\}$:

(1) if x and y are even, then $(x, y) \equiv (1, 2n - 2)$;
(2) if x and y are odd, then $(x, y) \equiv (1, 2n - 2)$;
(3) if x is odd and y is even, then $(x, y) \equiv (1, 2n - 2)$;
(4) if x is even and y is odd, then $(x, y) \equiv (0, 2n - 1)$;
(5) we also have $(0, 2n - 1) \not\equiv (2n - 2, 1)$.

For $x \in \{1, \ldots, 2n-2\}$, $N_{\mathrm{Ind}(G)}(x) = \{x-1, x+1\}$. By Lemma 10, $\{x-1, x+1\}$ is an interval of $G - x$ and hence $(2n, x-1) \equiv (2n, x+1)$. It follows that $2n \sim \{2i; 0 \leq i \leq n-1\}$ and $2n \sim \{2i+1; 0 \leq i \leq n-1\}$. By Lemma 10, $\{0, \ldots, 2n-3\} \cup \{2n\}$ is an interval of $G-(2n-1)$ because $N_{\mathrm{Ind}(G)}(2n-1) = \{2n-2\}$. In particular, $(2n, 2n-2) \equiv (1, 2n-2)$. Similarly, it follows from $N_{\mathrm{Ind}(G)}(0) = \{1\}$ that $(2n, 1) \equiv (2n-2, 1)$. Consequently, if $(1, 2n-2) \equiv (2n-2, 1)$, then $\{0, \ldots, 2n-1\}$ would be an interval of G. Thus, $(1, 2n-2) \not\equiv (2n-2, 1)$. By interchanging G and $G^\star$, assume that $1 \longrightarrow 2n-2$. For a contradiction, suppose that $(0, 2n-1) \equiv (2n-1, 0)$. We obtain that either $0 \cdots 2n-1$ and $G-(2n) = R_{2n}$ or $0 \longleftrightarrow 2n-1$ and $G-(2n) = (\overline{R_{2n}})^\star$. In both instances, $G-(2n)$ would be indecomposable, which contradicts the fact that G is critical. Therefore, $(0, 2n-1) \not\equiv (2n-1, 0)$. As $(0, 2n-1) \not\equiv (2n-2, 1)$, we have $0 \longrightarrow 2n-1$ and $G = W_{2n+1}$. □

The characterization of all the critical graphs follows from the four propositions above as well.

Theorem 22 (*Schmerl and Trotter [6]*). *Given an indecomposable graph $G = (V, A)$, with $|V| \geq 5$, G is critical if and only if G is isomorphic to one of the following graphs, where $m \geq 3$:* T_{2m-1}, U_{2m-1}, W_{2m-1}, H_{2m-1}, $\overline{H_{2m-1}}$, $\mathrm{Comp}(Q_{2m})$, $\mathrm{Comp}(R_{2m})$, Q_{2m}, $\overline{Q_{2m}}$, R_{2m} *or* $\overline{R_{2m}}$.

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